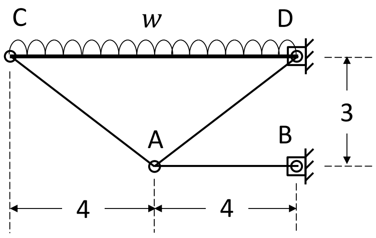
2014-2015 MM2MS3 Exam Solutions

1.

(a)

Labelling each position on the structure:



All dimensions in m

The distributed load, w, applied to the rigid beam, is distributed to the rest of the structure via points C and D. The distributed load can therefore be split equally between these two points as point loads, F, given as:

$$F = \frac{wL}{2} \quad (1)$$

[2 marks]

The structure can therefore be redrawn as:

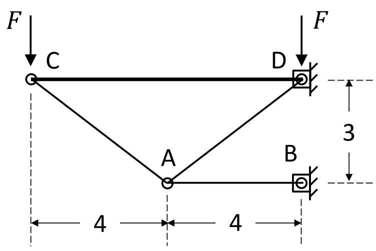


Fig 1.1

[2 marks]

Forces are in equilibrium; therefore, force polygons can be used. Drawing a force polygon for the force applied at C:

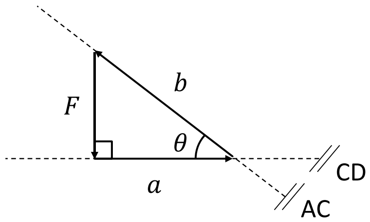
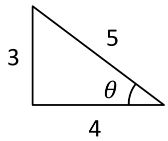


Fig 1.2

From geometry (see Fig 1.1) it can be seen that this is a 3-4-5 triangle, as follows:



$$\therefore \sin\theta = \frac{3}{5}$$

$$\therefore \theta = \sin^{-1}\left(\frac{3}{5}\right) = 36.87°$$

[1 mark]

Therefore, from Fig 1.2

$$\tan(36.87) = \frac{F}{a}$$

$$\therefore a = \frac{F}{\tan(36.87)} = \frac{4}{3}F$$

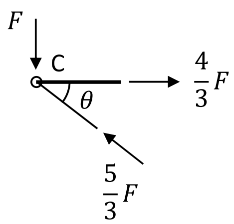
and,

$$\sin(36.87) = \frac{F}{b}$$

$$\therefore b = \frac{F}{\sin(36.87)} = \frac{5}{3}F$$

[1 mark]

Using the results from the force polygon to draw a free body diagram at joint C:



Critical load in member AC:

$$P_c{}^{AC} = \frac{\pi^2 EI}{L_{AC}{}^2} = \frac{5}{3}F$$

$$\therefore F = \frac{3\pi^2 EI}{5L_{AC}{}^2} = \frac{3\pi^2 EI}{5 \times 5{,}000^2}$$

$$\therefore F = \frac{3}{125{,}000{,}000}\pi^2 EI$$

[2 marks]

Free body diagram at joint A:

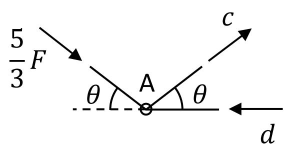


Fig 1.3

Vertical equilibrium gives:

$$\frac{5}{3}F\sin(36.87) = c\sin(36.87)$$

$$\therefore c = \frac{5}{3}F \qquad (2)$$

Horizontal equilibrium gives:

$$d = \frac{5}{3}F\cos(36.87) + c\cos(36.87)$$

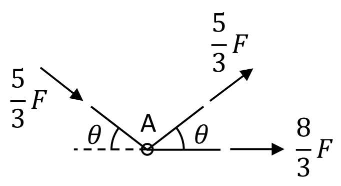
Substituting (2) into this gives:

$$\therefore d = \frac{4}{3}F + \frac{4}{3}F$$

$$\therefore d = \frac{8}{3}F$$

[1 mark]

Fig 1.3 can therefore be redrawn as:



Critical load in member AB:

$$P_c^{\ AB} = \frac{\pi^2 EI}{{L_{AB}}^2} = \frac{8}{3}F$$

$$\therefore F = \frac{3\pi^2 EI}{8{L_{AB}}^2} = \frac{3\pi^2 EI}{8 \times 4{,}000^2}$$

$$\therefore \boldsymbol{F} = \frac{\mathbf{3}}{\mathbf{128,000,000}}\boldsymbol{\pi^2 EI} \qquad (3)$$

[2 marks]

Critical load in member AD:

$$P_c^{\ AD} = \frac{\pi^2 EI}{{L_{AD}}^2} = \frac{5}{8}F$$

$$\therefore F = \frac{8\pi^2 EI}{5{L_{AD}}^2} = \frac{8\pi^2 EI}{5 \times 5{,}000^2} = \frac{8}{125{,}000{,}000}\pi^2 EI$$

[2 marks]

Therefore, since the value of F required to buckle member AB is smaller than that to buckle either member AC or member AD, it is this member (AB) that is most critical due to buckling.

[2 marks]

(b)

In member AB,

$$I = \frac{\pi D^4}{64} = \frac{\pi \times 40^4}{64} = 125{,}663.71 \text{ mm}^4$$

[2 marks]

Buckling load in member AB is given by (3). Substituting values into this equation gives:

$$\therefore F = \frac{3}{128{,}000{,}000}\pi^2 EI = \frac{3}{128{,}000{,}000} \times \pi^2 \times 200{,}000 \times 125{,}663.71 = 5{,}813.68\ \mathrm{N}$$

[3 marks]

Substituting this into equation (1) gives:

$$5{,}813.68 = \frac{w \times 8{,}000}{2}$$

$$\therefore w = 1.45342\ \frac{\mathrm{N}}{\mathrm{mm}} = 1{,}453.42\ \frac{\mathrm{N}}{\mathrm{m}}$$

[2 marks]

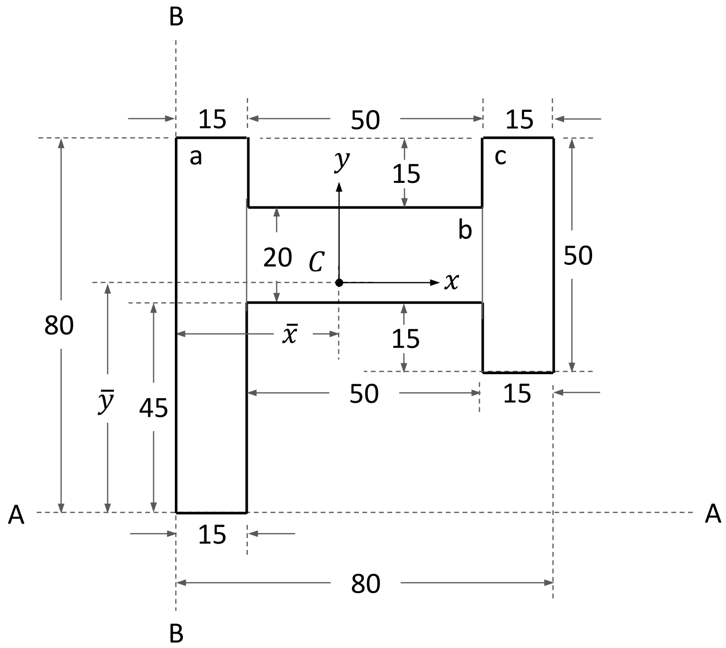
Applying a safety factor of 2 gives:

$$w = \frac{1{,}452.42\ \frac{\mathrm{N}}{\mathrm{m}}}{2} = \mathbf{726.71}\ \frac{\mathbf{N}}{\mathbf{m}}$$

[3 marks]

2.

(a)



Total area,

$$A = (15 \times 80)_a + (50 \times 20)_b + (15 \times 50)_c = 2950 \text{ mm}^2$$

[2 marks]

Taking moments about AA:

$$\bar{y} = \frac{(15 \times 80 \times 40)_a + (50 \times 20 \times 55)_b + (15 \times 50 \times 55)_c}{2950} = \mathbf{48.9\ mm}$$

[2 marks]

Similarly, taking moments about BB:

$$\bar{x} = \frac{(80 \times 15 \times 7.5)_a + (20 \times 50 \times 40)_b + (50 \times 15 \times 72.5)_c}{2950} = \mathbf{35.04\ mm}$$

[2 marks]

(b)

Using the Parallel Axis Theorem,

$$I_{x'} = (I_x + Ab^2)_a + (I_x + Ab^2)_b + (I_x + Ab^2)_c$$

$$= \left(\frac{15 \times 80^3}{12} + 15 \times 80 \times (40 - 48.9)^2\right) + \left(\frac{50 \times 20^3}{12} + 50 \times 20 \times (55 - 48.9)^2\right)$$

$$+ \left(\frac{15 \times 50^3}{12} + 15 \times 50 \times (55 - 48.9)^2\right)$$

$$= 989{,}752.83 \text{ mm}^4$$

[2 marks]

and,

$$I_{y'} = \left(I_y + Aa^2\right)_a + \left(I_y + Aa^2\right)_b + \left(I_y + Aa^2\right)_c$$

$$= \left(\frac{80 \times 15^3}{12} + 80 \times 15 \times (7.5 - 35.04)^2\right) + \left(\frac{20 \times 50^3}{12} + 20 \times 50 \times (40 - 35.04)^2\right)$$

$$+ \left(\frac{50 \times 15^3}{12} + 50 \times 15 \times (72.5 - 35.04)^2\right)$$

$$= 2{,}232{,}078.05 \text{ mm}^4$$

[2 marks]

Also,

$$I_{x'y'} = \left(I_{xy} + Aab\right)_a + \left(I_{xy} + Aab\right)_b + \left(I_{xy} + Aab\right)_c$$

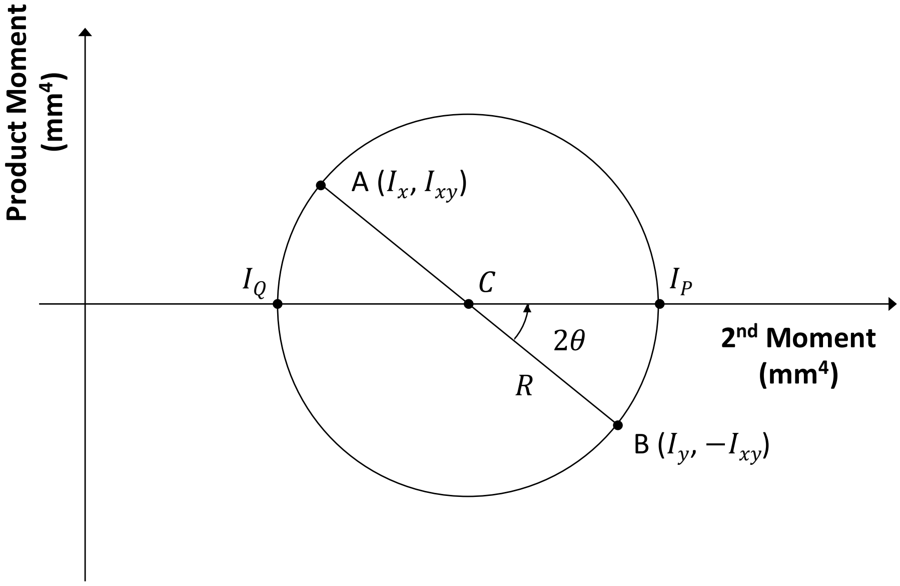
$$= \big(0 + 15 \times 80 \times (7.5 - 35.04) \times (40 - 48.9)\big) + \big(0 + 50 \times 20 \times (40 - 35.04) \times (55 - 48.9)\big)$$

$$+ \big(0 + 15 \times 50 \times (72.5 - 35.04) \times (55 - 48.9)\big)$$

$$= 495{,}762.7 \text{ mm}^4$$

[2 marks]

Mohr's Circle:



[2 marks]

$$Centre, C = \frac{I_{x'} + I_{y'}}{2} = \frac{989{,}752.83 + 2{,}232{,}078.05}{2} = 1{,}610{,}915.44 \text{ mm}^4$$

$$Radius, R = \sqrt{\left(\frac{I_{y'} - I_{x'}}{2}\right)^2 + {I_{x'y'}}^2} = \sqrt{\left(\frac{2{,}232{,}078.05 - 989{,}752.83}{2}\right)^2 + 495{,}762.7^2} = 794{,}747.22 \text{ mm}^4$$

[1 mark]

Therefore, the Principal 2nd Moments of Area are:

$$I_P = C + R = 1{,}610{,}915.44 + 794{,}747.22 = \mathbf{2,405,662.66\ mm^4}$$

[2 marks]

and,

$$I_Q = C - R = 1{,}610{,}915.44 - 794{,}747.22 = \mathbf{816,167.22\ mm^4}$$

[2 marks]

(c)

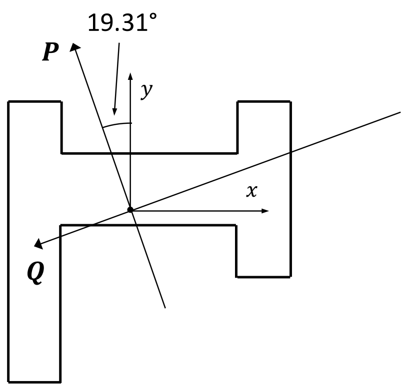
From the Mohr's circle above:

$$sin2\theta = \frac{I_{xy}}{R} = \frac{-495{,}762.7}{794{,}747.22}$$

$$\therefore \boldsymbol{\theta = -19.31°}$$

[3 marks]

Therefore, the Principal Axes are at $-19.31°$ (anti-clockwise) from the $x - y$ axes, as shown on the diagram below.



[3 marks]

3.

(a)

Figure Q3.1 shows an element of a straight beam, length δs, which bends to curvature R, due to an applied bending moment M. The angle subtended by the element of beam is $\delta\phi$, also equal to the change in slope of the beam over δs.

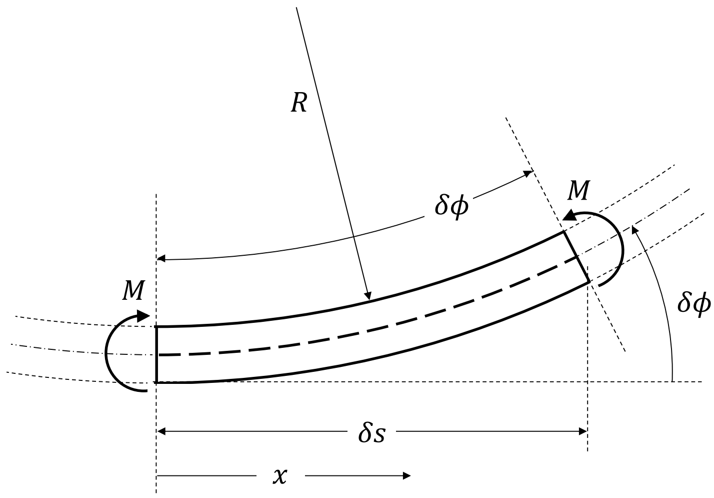


Fig Q3.1 Element of a Beam

[1 mark]

Therefore, the strain energy (work done) for the element, δU, is given by (area under the curve in Fig Q3.2):

$$\delta U = \frac{1}{2} M \delta\phi \qquad (1)$$

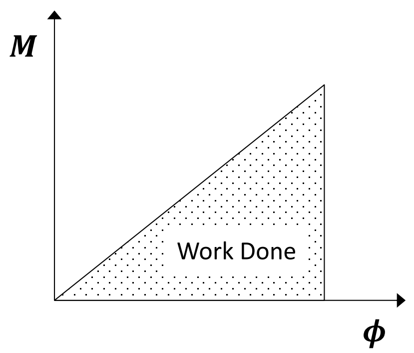


Fig Q3.2 Plot of Strain Energy in a Beam

[1 mark]

Equation of an arc:

$$\delta s = R\delta\phi \qquad (2)$$

and Beam Bending equation:

$$\frac{M}{I} = \frac{E}{R} \qquad (3)$$

[1 mark]

Therefore, rearranging (3) for R and substituting this into (2):

$$\delta s = \frac{EI}{M}\delta\phi$$

$$\therefore\ \delta\phi = \frac{M}{EI}\delta s$$

Substituting this into (1) gives:

$$\delta U = \frac{M^2}{2EI}\delta s \qquad (4)$$

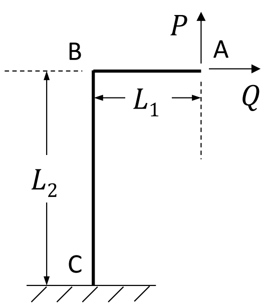
Thus, for a beam of length, L, integrating (4) across this length gives:

$$\boldsymbol{U = \int_0^L \frac{M^2}{2EI}\delta s}$$

[2 marks]

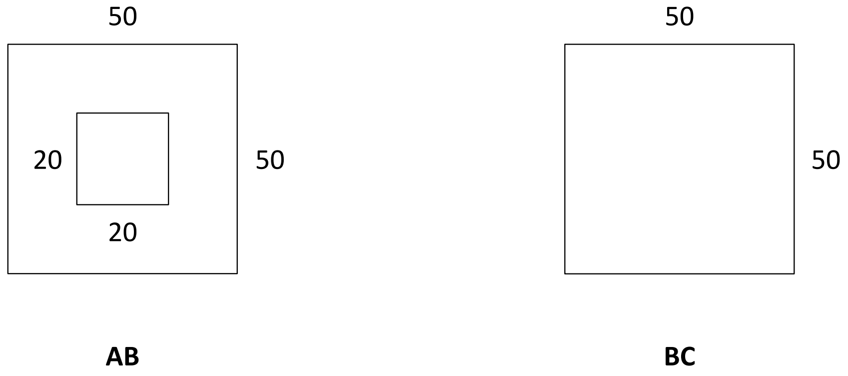
(b)

Adding dummy load, Q, and labelling each corner:



[2 marks]

Cross sections of the two section of the beam (AB and BC):



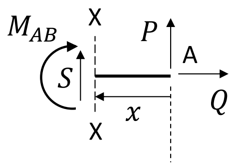
$$I_{AB} = \left(\frac{b_{AB}{d_{AB}}^3}{12}\right)_{outer} - \left(\frac{b_{AB}{d_{AB}}^3}{12}\right)_{inner} = \frac{50\times 50^3}{12} - \frac{20\times 20^3}{12} = \frac{50^4}{12} - \frac{20^4}{12} = 507{,}500\ \mathrm{mm}^4$$

[1 mark]

$$I_{BC} = \frac{b_{BC}{d_{BC}}^3}{12} = \frac{50\times 50^3}{12} = \frac{50^4}{12} = 520{,}833.33\ \mathrm{mm}^4$$

[1 mark]

Free-body diagram for Section AB:



[2 marks]

Taking moments about X-X:

$$M_{AB} = Px$$

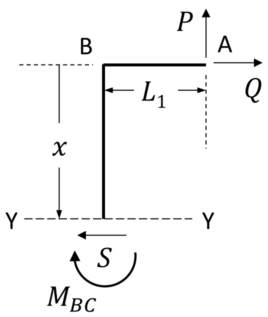
[1 mark]

Substituting this into the equation for strain energy for a beam under bending (see section (a)):

$$U_{AB} = \int \frac{{M_{AB}}^2}{2EI_{AB}}\delta s = \int_0^{L_1} \frac{(Px)^2}{2EI_{AB}}\delta x = \frac{P^2}{2EI_{AB}}\int_0^{L_1} x^2\,\delta x = \frac{P^2}{2EI_{AB}}\left[\frac{x^3}{3}\right]_0^{L_1} = \frac{P^2}{2EI_{AB}}\left(\frac{{L_1}^3}{3} - 0\right) = \frac{P^2{L_1}^3}{6EI_{AB}}$$

[2 marks]

Free-body diagram for Section BC:



[2 marks]

Taking moments about X-X:

$$M_{BC} + Qx = PL_1$$

$$\therefore M_{BC} = PL_1 - Qx$$

[1 mark]

Substituting this into the equation for strain energy for a beam under bending (see section(a)):

$$U_{BC} = \int \frac{{M_{AB}}^2}{2EI_{AB}} \delta s = \int_0^{L_2} \frac{(PL_1 - Qx)^2}{2EI_{BC}} \delta x = \frac{1}{2EI_{BC}} \int_0^{L_2} \left(P^2{L_1}^2 - 2PQL_1x + Q^2x^2\right) \delta x$$

$$= \frac{1}{2EI_{BC}} \left[P^2{L_1}^2x - PQL_1x^2 + \frac{Q^2x^3}{3}\right]_0^{L_2} = \frac{1}{2EI_{BC}} \left(\left(P^2{L_1}^2L_2 - PQL_1{L_2}^2 + \frac{Q^2{L_2}^3}{3}\right) - (0 - 0 + 0)\right)$$

$$= \frac{1}{2EI_{BC}} \left(P^2{L_1}^2L_2 - PQL_1{L_2}^2 + \frac{Q^2{L_2}^3}{3}\right)$$

[2 marks]

$$U_{total} = U_{AB} + U_{BC}$$

$$\therefore U_{total} = \frac{P^2{L_1}^3}{6EI_{AB}} + \frac{1}{2EI_{BC}} \left(P^2{L_1}^2L_2 - PQL_1{L_2}^2 + \frac{Q^2{L_2}^3}{3}\right) \qquad (5)$$

[2 marks]

Calculation of horizontal deflection at A, u_{H_A}:

Integrating (5) with respect to Q:

$$u_{H_A} = \frac{\partial U_{total}}{\partial Q} = \frac{1}{2EI_{BC}} \left(-PL_1{L_2}^2 + \frac{2Q{L_2}^3}{3}\right)$$

[2 marks]

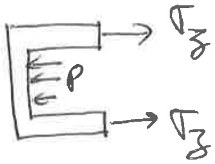
Setting dummy load, Q, to zero, gives:

$$u_{H_A} = \frac{-PL_1{L_2}^2}{2EI_{BC}} = \frac{-1000 \times 250 \times 500^2}{2 \times 70{,}000 \times 520{,}833.33} = \mathbf{-0.857\ mm}$$

[2 marks]

4.

(a) Axial stress due to internal pressure:



$$\sigma_z \times \pi \times (R_o^2 - R_i^2) = p \times \pi R_i^2$$

$$\therefore \sigma_z = \frac{300^2}{305^2 - 300^2} \times 1 \text{ MN/m}^2 = 29.752 \text{ MN/m}^2$$

Axial stress due temperature difference:

$$\begin{cases} \varepsilon_\theta = \frac{1}{E}\left[\sigma_\theta - \nu(\sigma_r + \sigma_z)\right] + \alpha T \\ \varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_r + \sigma_\theta)\right] + \alpha T \end{cases}$$

The cylinder is unrestrained $\rightarrow \sigma_r = 0$

Compatibility condition $\longrightarrow \varepsilon_\theta = \varepsilon_z = 0$

$$\therefore \begin{cases} 0 = \frac{1}{E}\left[\sigma_\theta - \nu\sigma_z\right] + \alpha T \\ 0 = \frac{1}{E}\left[\sigma_z - \nu\sigma_\theta\right] + \alpha T \end{cases}$$

$\rightarrow$ This is only true if $\sigma_z = \sigma_\theta$

$$\therefore \sigma_\theta = \sigma_z = \frac{-E\alpha T}{(1-\nu)}$$

$$= \frac{-E\alpha}{(1-\nu)} \times \Delta T \times \frac{y}{t}$$

$$= \frac{-200\times10^3 \times 12\times10^{-6} \times 4}{0.7\times5}\, y$$

$$= -2.743\, y \text{ MN/m}^2$$

$$\sigma_{z_i} = -2.743 \times 2.5 = -6.857 \text{ MN/m}^2$$

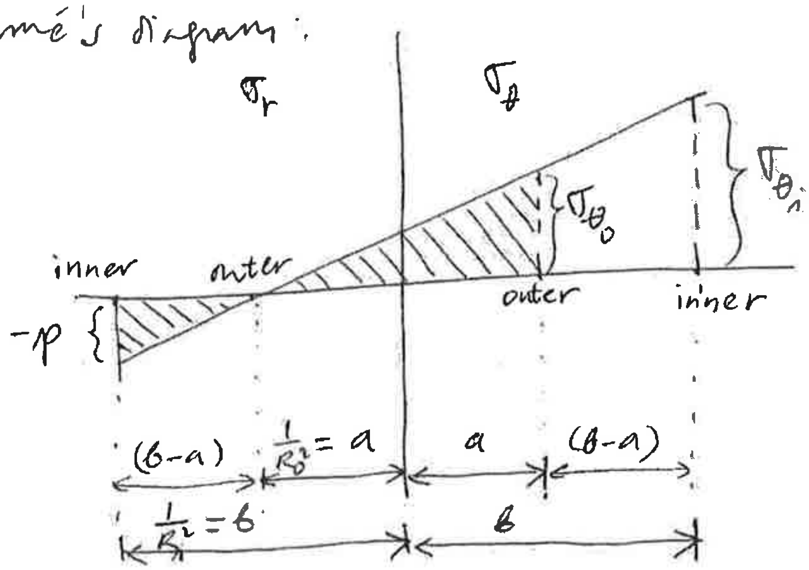
$$\sigma_{z_o} = +6.857 \text{ MN/m}^2$$

The total axial stresses:

$$\sigma_{z_i} = 29.752 - 6.857 = \underline{\underline{22.895 \text{ MN/m}^2}}$$

$$\sigma_{z_o} = 29.752 + 6.857 = \underline{\underline{36.609 \text{ MN/m}^2}}$$

(b) Lamé's diagram:



$R_i = 300$ mm

$R_o = 305$ mm

Hoop stresses due to internal pressure:

$$\sigma_{\theta_o} = \frac{2a}{(b-a)} \times p = \frac{2 \times \frac{1}{R_o^2}}{\frac{1}{R_i^2} - \frac{1}{R_o^2}} \times 1 \text{ MN/m}^2 = 59.504 \text{ MN/m}^2$$

$$\sigma_{\theta_i} = \frac{2a + (b-a)}{(b-a)} \times p = 60.504 \text{ MN/m}^2$$

Hoop stresses due to temperature difference:

$$\sigma_{\theta_i} = -6.857 \text{ MN/m}^2 \quad ; \quad \sigma_{\theta_o} = +6.857 \text{ MN/m}^2$$

Total stresses:

At inside:

$\sigma_r = -1$ MN/m²

$\sigma_\theta = 60.504 - 6.857 = \underline{\underline{53.641}}$ MN/m²

$\sigma_z = 22.895$ MN/m²

At outside:

$\sigma_r = 0$

$\sigma_\theta = 59.504 + 6.857 = \underline{\underline{66.361}}$ MN/m²

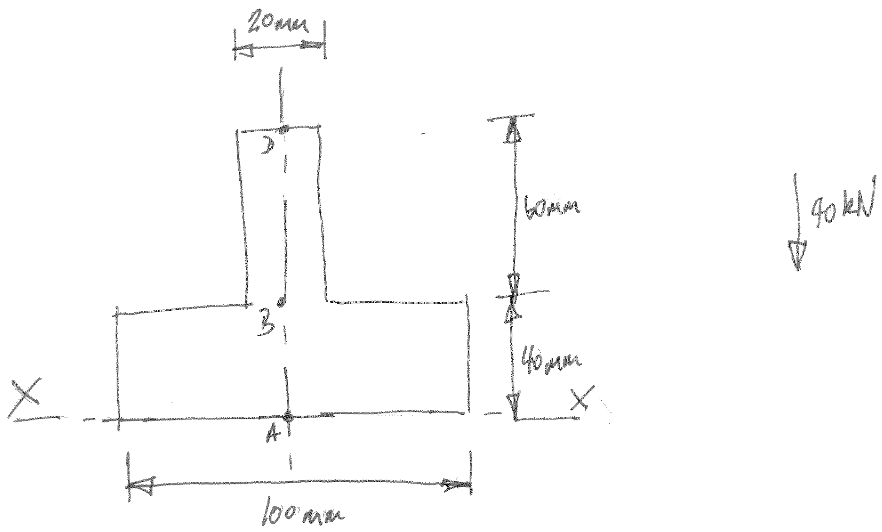
$\sigma_z = 36.609$ MN/m²

(c) Max shear stress is at outside:

$$\tau_{max} = \frac{66.361 - 0}{2} = 33.182 \text{ MN/m}^2$$

$$\text{Factor of Safety} = \frac{~~200~~\ 250}{66.361} = ~~3~~\ \underline{\underline{3.77}}$$

5.



a) Determine the distance of the centroid, C, from the base x–x

First moment of area:

$$\bar{y} = \frac{\Sigma A\bar{y}}{A_T} = \frac{((40\times100)\times20)+((60\times20)\times70)}{(40\times100)+(60\times20)}$$

$$= \frac{(4000\times20)+(1200\times70)}{5200}$$

$$= \underline{31.5\,mm}$$

b) Calculate the second moment of area:

$$I = \Sigma\left(\frac{BD^3}{12}+A\bar{y}^2\right) = \left(\frac{100\times40^3}{12}+(40\times100)\times(31.5-20)^2\right) + \left(\frac{20\times60^3}{12}+(20\times60)\times(70-31.5)^2\right)$$

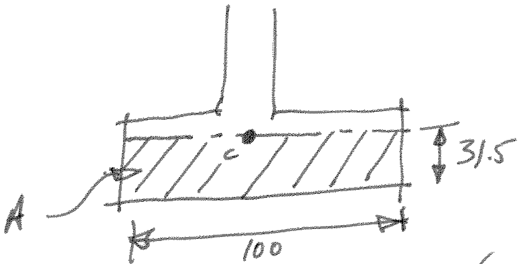
$$= (533333.3+529000)+(360000+1778700)$$

$$= \underline{3201033\,mm^4}$$

c) A and D are free surfaces, therefore $\tau = 0$.

$$\tau = \frac{SA\bar{y}}{IZ}$$

At point C:



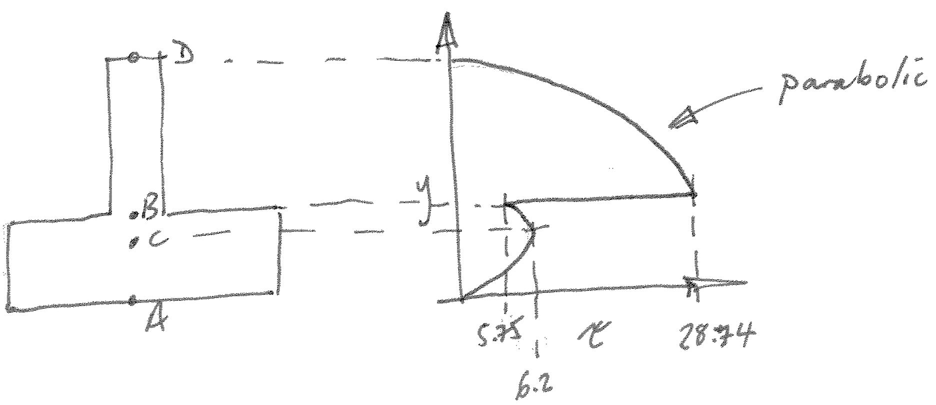
$$\tau_C = \frac{40000 \times (31.5 \times 100) \times (31.5/2)}{3201033 \times 100} = \frac{1.9845 \times 10^9}{3.201 \times 10^8} = 6.2\,MPa.$$

At point B, two values due to section change

$$\tau_{B1} = \frac{40,000 \times (40 \times 100) \times (31.5-20)}{3201033 \times 100} = \frac{1.84 \times 10^9}{3.201 \times 10^8} = 5.75\,MPa.$$

$$\tau_{B2} = \frac{40000 \times (40 \times 100) \times (31.5-20)}{3201033 \times 20} = \frac{1.84 \times 10^9}{6.40 \times 10^7} = 28.74\,MPa.$$

d)



6.

(a)

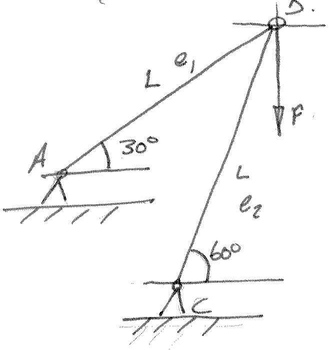
$$[k_e] = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$$

where ,

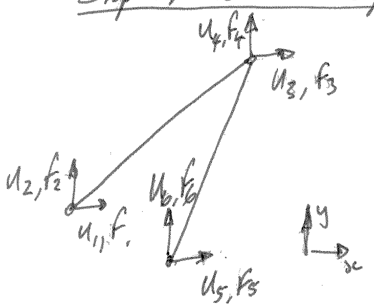
$$k = \frac{AE}{L}$$

(b)

Geometry



Disp & Force coord sys



$AE = 200\,MN$

$L = 1\,m$

element stiffness matrix

i) $[k_{e1}]$ $\angle = 30°$

$$[k_{e1}] = 200\times10^6 \begin{bmatrix} 0.75 & \sqrt{3}/4 & -0.75 & -\sqrt{3}/4 \\ \sqrt{3}/4 & 0.25 & -\sqrt{3}/4 & -0.25 \\ -0.75 & -\sqrt{3}/4 & 0.75 & \sqrt{3}/4 \\ -\sqrt{3}/4 & -0.25 & \sqrt{3}/4 & 0.25 \end{bmatrix}$$

$[k_{e2}]$ $\angle = 240°$

$$[k_{e2}] = 200\times10^6 \begin{bmatrix} 0.25 & \sqrt{3}/4 & 0.25 & -\sqrt{3}/4 \\ \sqrt{3}/4 & 0.75 & -\sqrt{3}/4 & -0.75 \\ -0.25 & -\sqrt{3}/4 & 0.25 & \sqrt{3}/4 \\ -\sqrt{3}/4 & -0.75 & \sqrt{3}/4 & 0.75 \end{bmatrix}$$

$$[k_g] = 200\times10^6 \begin{bmatrix} 0.75 & \sqrt{3}/4 & -0.75 & -\sqrt{3}/4 & 0 & 0 \\ \sqrt{3}/4 & 0.25 & -\sqrt{3}/4 & -0.25 & 0 & 0 \\ -0.75 & -\sqrt{3}/4 & 1 & \sqrt{3}/2 & 0.25 & -\sqrt{3}/4 \\ -\sqrt{3}/4 & -0.25 & \sqrt{3}/2 & 1 & -\sqrt{3}/4 & -0.75 \\ 0 & 0 & -0.25 & -\sqrt{3}/4 & 0.25 & \sqrt{3}/4 \\ 0 & 0 & -\sqrt{3}/4 & -0.75 & \sqrt{3}/4 & 0.75 \end{bmatrix}$$

global 'structure' stiffness matrix

(c)

$$\{F\} = [k]\{u\}$$

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix} = [k] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix}$$

$$\begin{Bmatrix} 0 \\ -20{,}000 \end{Bmatrix} = 200\times10^6 \begin{bmatrix} 1 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix}$$

$$200\times10^6 u_3 + \sqrt{3}\times1\times10^8 u_4 = 0 \Rightarrow u_4 = \frac{-200\times10^6 u_3}{\sqrt{3}\times1\times10^8} = -1.15u_3$$

$$\sqrt{3}\times1\times10^8 u_3 + 200\times10^6 u_4 = -20000$$

$$\left(\sqrt{3}\times10^8 + (200\times10^6)\times-1.15\right)u_3 = -20000$$

$$\Rightarrow -5.77\times10^7 u_3 = -20000$$

$$\Rightarrow u_3 = \underline{3.47\times10^{-4}\,m} = 0.35\,mm$$

$$\Rightarrow u_4 = \frac{-6.93\times10^4}{\sqrt{3}\times1\times10^8}$$

$$= \underline{-4.00\times10^{-4}\,m} = \underline{-0.4\,mm}$$

(d)

$$f_1 = 200\times10^6 \times \left(-0.75u_3 - \sqrt{3}/4\,u_4\right)$$

$$= 200\times10^6 \times \left(-0.75\times3.47\times10^{-4} - \sqrt{3}/4\times-4\times10^{-4}\right)$$

$$= 200\times10^6 \times \left(-2.602\times10^{-4} + 1.73\times10^{-4}\right)$$

$$= 200\times10^6 \times -8.7\times10^{-5}$$

$$= \underline{-17.4\,kN}$$

$$f_2 = 200\times10^6 \times \left(-\sqrt{3}/4\, u_3 - 0.25\, u_4\right)$$

$$= 200\times10^6 \times \left(-1.5\times10^{-4} + 1\times10^{-4}\right)$$

$$= 200\times10^6 \times -0.5\times10^{-4}.$$

$$= -10\,kN$$

$$f_5 = 200\times10^6 \times \left(-0.25 u_3 - \sqrt{3}/4\, u_4\right)$$

$$= 200\times10^6 \times \left(-8.68\times10^{-4} + 1.73\times10^{-4}\right)$$

$$= 200\times10^6 \times 8.625\times10^{-5}$$

$$= 17.25\,kN$$

$$f_6 = 200\times10^6 \times \left(-\sqrt{3}/4\, u_3 - 0.75\, u_4\right)$$

$$= 200\times10^6 \times \left(-1.5\times10^{-4} + 3\times10^{-4}\right)$$

$$= 200\times10^6 \times 1.5\times10^{-4}$$

$$= 30\,kN$$